The relationship between singular integrals and Lebesgue integrals

Question: Let μ be the Lebesgue measure on $\mathbb{R}$ and let f be a continuous functions on $\mathbb{R}$. When restricted to $[a,b]$, it is known that $\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathrm{d}\mu$, where the left hand side is the Riemann integral and the right hand side is the Lebesgue integral. Assume that $\int_0^{+\infty} f\,\mathrm{d}t$ exists, can we claim that $\int_0^{+\infty} f\mathrm{d}t = \int_{[0,+\infty)} f\,\mathrm{d}\mu$?

Answer: If we assume certain other things such as f is non-negative, or $\int_0^{+\infty} |f|\,\mathrm{d}t < \infty$, etc, yes. In general, no.

First, note that the Riemann integral $\int_0^{+\infty} f\,\mathrm{d}t$ cannot be defined directly in the sense that the Darboux upper sum equals the Darboux lower sum (why?). Instead, this Riemann integral is defined as

$$\int_0^{+\infty} f\,\mathrm{d}t = \lim_{x\to+\infty}\int_0^x f\,\mathrm{d}t.$$

On each $[0,x]$, by the result of Problem 2 in Homework 4, we have

$$\int_0^x f\,\mathrm{d}t = \int_{[0,x]} f\,\mathrm{d}\mu.$$

Thus

$$\begin{aligned}\int_0^{+\infty} f\,\mathrm{d}t &= \lim_{x\to+\infty}\int_0^x f\,\mathrm{d}t\\ &= \lim_{x\to+\infty}\int_{[0,x]} f\,\mathrm{d}\mu\\ &= \lim_{x\to+\infty}\int_{[0,+\infty)} \chi_{[0,x]}\cdot f\,\mathrm{d}\mu\end{aligned}$$

Also, note that

$$\int_{[0,+\infty)} f\,\mathrm{d}\mu = \int_{[0,+\infty)} \lim_{x\to+\infty}[\chi_{[0,x)}\cdot f]\,\mathrm{d}\mu.$$

The original question is just equivalent to whether $\lim_{x\to+\infty}\int_{[0,+\infty)}\chi_{[0,x]}\cdot f\,\mathrm{d}\mu$ equals $\int_{[0,+\infty)}\lim_{x\to+\infty}[\chi_{[0,x)}\cdot f]\,\mathrm{d}\mu$.

When f is positive, it is clear that $\chi_{[0,x_1]}\cdot f \geq \chi_{[0,x_2]}\cdot f$ if $x_1 \geq x_2$. By Lebesgue Monotone Convergence Theorem (not a direct use of it, because we have $x \to +\infty$ along the interval $(0,+\infty)$ instead of along a sequence. But we can still manage to let it work. Why?), it follows that

$$\lim_{x\to+\infty}\int_{[0,+\infty)}\chi_{[0,x]}\cdot f\,\mathrm{d}\mu=\int_{[0,+\infty)}\lim_{x\to+\infty}[\chi_{[0,x)}\cdot f]\,\mathrm{d}\mu.$$

So far, we have shown that the answer to the original problem might be yes under some extra conditions. Now, we will show that, without assuming anything more than the continuity of f and the existence of $\int_0^{+\infty} f\,\mathrm{d}t$, the answer to the original problem might be no.

Example 1: Choose a continuous function f such that

$$f(0)=f(1)=f(2)=f(3)=\cdots=0,$$

$$\int_0^1 f(t)\,\mathrm{d}t=\int_2^3 f(t)\,\mathrm{d}t=\int_4^5 f(t)\,\mathrm{d}t=\cdots=2,$$
$$\int_1^2 f(t)\,\mathrm{d}t=\int_3^4 f(t)\,\mathrm{d}t=\int_5^6 f(t)\,\mathrm{d}t=\cdots=-1,$$

f is positive on $[0,1]\sqcup[2,3]\sqcup[4,5]\sqcup\cdots$ and is negative on $[1,2]\sqcup[3,4]\sqcup[5,6]\sqcup\cdots$.

For this f, one can check that $\int_0^{+\infty} f\,\mathrm{d}t=+\infty$. As $\int_{[0,+\infty)} f^+\,\mathrm{d}\mu=\int_{[0,+\infty)} f^-\,\mathrm{d}\mu=+\infty$, according to the definition, the Lebesgue integration $\int_{[0,+\infty)} f\,\mathrm{d}\mu$ does not exist at all.

The next example is one such that $\int_0^{+\infty} f\,dt$ exists and is finite, but $\int_{[0,+\infty)} f\,d\mu$ does not exist.

Example 2: According to the known facts of series, we can find positive sequences $\{a_n\}$ and $\{b_n\}$, such that

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=0,$$

$$\sum_{n=0}^{\infty}a_n=\sum_{n=0}^{\infty}b_n=+\infty,$$

and the series

$$a_0-b_1+a_1-b_1+a_2-b_2+a_3-b_3\cdots$$

converge to certain $L\in(-\infty,+\infty)$.

Similar to Example 1, we choose a continuous function f such that

$$f(0) = f(1) = f(2) = f(3) = \cdots = 0,$$

$$\int_{2n}^{2n+1} f(t)\,\mathrm{d}t = a_n \quad \forall\, n \in \mathbb{N}_{\geq 0},$$

$$\int_{2n+1}^{2n+2} f(t)\,\mathrm{d}t = -b_n \quad \forall\, n \in \mathbb{N}_{\geq 0}.$$

f is positive on $[0,1] \sqcup [2,3] \sqcup [4,5] \sqcup \cdots$ and is negative on $[1,2] \sqcup [3,4] \sqcup [5,6] \sqcup \cdots$.

Consider

$$F_x = \int_0^x f\,\mathrm{d}t.$$

One can prove that $\{F_x\}_{x\in\mathbb{R}_{\geq 0}}$ is Cauchy in the sense that for any $\epsilon > 0$, there exists $M \in \mathbb{R}_{\geq 0}$, such that for any $x, y \in (M, +\infty)$, we have $|F_x - F_y| < \epsilon$. Note that we are not talking about a Cauchy sequence here. The proof will need such requirement that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. During the proof, one should also note that the partial sum corresponding to the series $a_0 - b_1 + a_1 - b_1 + a_2 - b_2 + a_3 - b_3 \cdots$ is a Cauchy sequence. That proof is left as an exercise.

Now, as $\{F_x\}_{x\in\mathbb{R}_{\geq 0}}$ is Cauchy (not a Cauchy sequence though), $\lim_{x\to\infty} F_x$ exists (why?). Thus $\int_0^{+\infty} f\,\mathrm{d}t$ exists and is exactly L.

The Lebesgue integration $\int_{[0,+\infty)} f\,\mathrm{d}\mu$, however, does not exist at all because both $\int_{[0,+\infty)} f^+\,\mathrm{d}\mu$ and $\int_{[0,+\infty)} f^-\,\mathrm{d}\mu$ are $+\infty$ (noting that $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = +\infty$).